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Numerical methods for solving singular integral equations with Cauchy-type kernels

T. Ramanna, R. Molotsi, J. Malele, C. Khoza, G. Sibelo, and Dr. M. Nchabeleng (Presenter)

Overview

1. Study Problem
2. Analytical Solution
3. Approximate Analytical Solution
4. Finite Difference Method
5. Integral Approximation Approach
6. Conclusion

Study Problem

Study Problem

Cauchy-type singular integral equation

- Consider the problem of solving the singular integral equation given by

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \frac{\varphi_\xi(\xi)}{\xi - x}\mathrm{d}\xi\right) = 1, \qquad 0 \leqslant x \leqslant 1, \tag{1}$$

  subject to the boundary conditions

$$\varphi(0) = \frac{3\pi}{8},\ \varphi(1) = 0 \textit{ and } \varphi_x(0) = 0. \tag{2a-c}$$

- We want to solve (1) subject to (2a-c) both analytically and numerically.

Study Problem

We approached the solution of this problem in 4 ways:

- Analytical Solution
- Approximate Analytical Solution
  Using Chebyshev polynomials
- Finite Difference Method
  Using central difference approximation
- Integral Approximation Approach
  Used the left-end point rule for the integral

Analytical Solution

Analytical Solution

- Integrating our problem w.r.t x we obtain the following characteristic integral equation

$$\int_0^1 \frac{\varphi_\xi(\xi)d\xi}{\xi - x} = \pi x + A. \tag{3}$$

- Using the standard inversion formula from [1] we obtain the inverse of (3) as

$$\varphi_x(x) = \frac{c}{\sqrt{x(1-x)}} - \frac{1}{\pi^2\sqrt{x(1-x)}}\int_0^1 \frac{\sqrt{\xi(1-\xi)}}{\xi - x}(\pi\xi + A)d\xi. \tag{4}$$

- We solved the integral on the LHS of (3) using (4) to obtain

$$\varphi(x) = C\sin^{-1}(2x-1) - \left(\frac{x}{2} + \frac{1}{4} + \frac{A}{\pi}\right)\sqrt{x(1-x)} + B. \tag{5}$$

- We then used the conditions in (2a-c) to solve for the constants.

Analytical Solution

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- Solving for the constants we obtained

$$A = -\pi, \quad B = \frac{3\pi}{16} \quad \text{and} \quad C = -\frac{3}{8}. \tag{6}$$

- The analytical solution is therefore

$$\varphi(x) = -\frac{3}{8}\sin^{-1}(2x-1) - \left(\frac{x}{2} - \frac{3}{4}\right)\sqrt{x(1-x)} + \frac{3\pi}{16}. \tag{7}$$

Approximate Analytical Solution

Approximate Analytical Solution

- In this section solve equation (1) using Chebyshev's polynomials.
- First we transform equation (1) by letting,

$$\varphi(x) = H(y) \quad \text{where} \quad y = 2x - 1. \tag{8}$$

- This gives us

$$\frac{d}{dy}\left(\int_{-1}^{1} \frac{H_s(s)}{s-y} ds\right) = \frac{\pi}{4}, \tag{9}$$

subject to $H(-1) = \frac{3\pi}{8}$, $H(1) = 0$ and $H_y(-1) = 0$.
- For mathematical convenience we let $\phi(s) = H_s(s)$ and thus

$$\frac{d}{dy}\left(\int_{-1}^{1} \frac{\phi(s)}{s-y} ds\right) = \frac{\pi}{4}, \tag{10}$$

Approximate Analytical Solution

- Let the unknown function ϕ in (10) be approximated by the polynomial function $\phi_n(x)$

$$\phi_n(x) = w(x) \sum_{i=0}^{n} \beta_i T_i(x), \tag{11}$$

where $\beta_i, i = 0, 1, 2, 3, ..., n$ are unknown coefficients, $w(x)$ is the weight function

$$w(x) = \frac{1}{\sqrt{1 - x^2}}, \tag{12}$$

and the Chebyshev polynomial is $T_i = \cos[i \cos^{-1}(x)]$.

Approximate Analytical Solution

This gives us

$$\frac{d}{dy}\Big(\sum_{i=0}^{n}\beta_i\gamma_i(y)\Big) = \frac{\pi}{4}, \tag{13}$$

where

$$\gamma_i(y) = \int_{-1}^{1}\frac{w(s)T_i(s)}{s-y}dy = \int_{-1}^{1}\frac{T_i(s)}{\sqrt{1-s^2}(s-y)}dy. \tag{14}$$

Let x_k be a zero of the Chebyshev polynomials of the 2^{nd} kind

$$U_{i-1}(x) = \frac{\sin(i\cos^{-1}(x))}{\sin(\cos^{-1}(x))} \tag{15}$$

then,

$$x_k = cos\Big(\frac{k\pi}{i+1}\Big). \tag{16}$$

Approximate Analytical Solution

From [2] we know that

$$\int_{-1}^{1} \frac{T_i(s)}{\sqrt{1-s^2}(s-y)} dy = \pi U_{i-1}(y) \tag{17}$$

Therefore we get

$$\frac{d}{dy}\left(\sum_{i=0}^{n} \beta_i U_{i-1}(y)\right) = \frac{1}{4}. \tag{18}$$

Setting $n = 5$ we get,

$$\frac{d}{dy}\left[\beta_1 + 2\beta_2 y + \beta_3(4y^2 - 1) + \beta_4(8y^3 - 4y + \beta_5(16y^4 - 12y^2 + 1))\right]. \tag{19}$$

Approximate Analytical Solution

Equating the coefficients by their powers we found $\beta_2 = \frac{1}{8}$ and $\beta_3 = \beta_4 = \beta_5 = 0$. Thus for $n = 5$, we have

$$\phi_5(y) = \frac{1}{\sqrt{1-y^2}}\left[\beta_0 + \beta_1 y + \frac{1}{8}(2y^2 - 1)\right] = H_y(y). \tag{20}$$

Integrating (20) w.r.t y we obtain

$$H(y) = \beta_0 \sin^{-1}(y) - \frac{y(1-y^2)}{8} - \beta_1\sqrt{1-y^2} + C. \tag{21}$$

Approximate Analytical Solution

Using the boundary conditions we obtain the rest of the constants, $C = \frac{3\pi}{16}$, $\beta_0 = -\frac{3}{8}$ and $\beta_1 = -\frac{1}{4}$. Thus we have,

$$H(y) = -\frac{3}{8}\sin^{-1}(y) - \frac{y(1-y^2)}{8} + \frac{1}{4}\sqrt{1-y^2} + \frac{3\pi}{16}. \tag{22}$$

Substituting back (8) and rearranging the result we obtain,

$$\varphi(x) = -\frac{3}{8}\sin^{-1}(2x-1) - \left(\frac{x}{2} - \frac{3}{4}\right)\sqrt{(x(1-x)} + \frac{3\pi}{16}. \tag{23}$$

which is the same as the analytical solution.

Approximate Analytical Solution

It should be noted that the approximate analytical solution is equal to the analytical solution when the forcing function is linear.

Finite Difference Method

Finite Difference Method

We partition the interval of integration[0,1] into n equally spaced subintervals $[\xi_i, \xi_{i+1}]$ of length $h = \frac{1}{n}$, where $0 \leq i \leq n-1$.

Let $P(x)$ be

$$P(x) = \int_0^1 \frac{\varphi_\xi(\xi) d\xi}{\xi - x}, \tag{24}$$

then (1) becomes

$$\frac{dP}{dx} = \pi. \tag{25}$$

Finite Difference Method

Using the central finite difference approximation on (25) we get,

$$\frac{dP}{d\xi} = \frac{P_{i+\frac{1}{2}} - P_{i-\frac{1}{2}}}{\xi_{i+\frac{1}{2}} - \xi_{i-\frac{1}{2}}} = \pi, \qquad 1 \leq i \leq n-1 \tag{26}$$

where $P_{i\pm\frac{1}{2}}$ and $\xi_{i\pm\frac{1}{2}}$ are respectively given by

$$P_{i\pm\frac{1}{2}} = \int_0^1 \frac{\varphi_\xi(\xi)d\xi}{\xi - \xi_{i\pm\frac{1}{2}}} \tag{27}$$

$$\xi_{i\pm\frac{1}{2}} = \frac{i \pm \frac{1}{2}}{n} \tag{28}$$

Finite Difference Method

Assuming that the slope is constant in each subinterval and approximating $\varphi_\xi(\xi)$ using forward difference, we obtain

$$P_{i\pm\frac{1}{2}} = \sum_{j=0}^{n-1} \left(\frac{\varphi_{j+1} - \varphi_j}{\xi_{j+1} - \xi j} \right) \int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - \xi_{i\pm\frac{1}{2}}} d\xi. \tag{29}$$

Substituting (29) into (26), we obtain

$$\sum_{j=0}^{n-1} \frac{\varphi_{j+1} - \varphi j}{(\xi_{i+\frac{1}{2}} - \xi_{i-\frac{1}{2}})(\xi_{j+1} - \xi_j)} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - \xi_{i+\frac{1}{2}}} d\xi - \sum_{j=0}^{n-1} \frac{\varphi_{j+1} - \varphi_j}{(\xi_{i+\frac{1}{2}} - \xi_{i-\frac{1}{2}})(\xi_{j+1} - \xi_j)} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - \xi_{i-\frac{1}{2}}} d\xi \tag{30}$$

Finite Difference Method

Integrating the integral terms we get

$$\int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - \xi_{i\pm\frac{1}{2}}} d\xi = \ln\left|\frac{\xi_{j+1} - \xi_{i\pm\frac{1}{2}}}{\xi_j - \xi_{i\pm\frac{1}{2}}}\right|. \tag{31}$$

Substituting (31), (30) becomes

$$\sum_{j=0}^{n-1} (\varphi_{j+1} - \varphi_j) m_{i,j} = \frac{\pi}{n^2}, \tag{32}$$

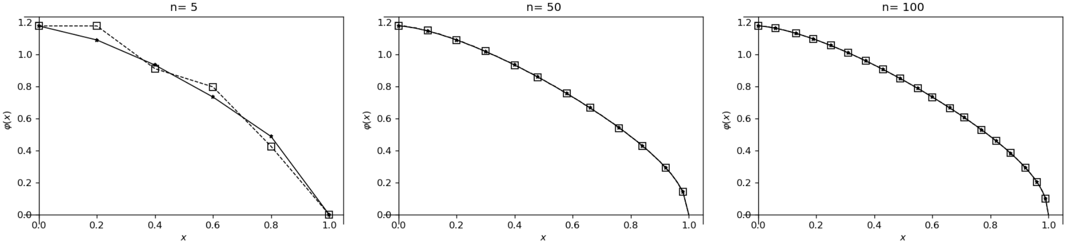
where $\xi_{i+1/2} - \xi_{i-1/2} = \xi_{j+1} - \xi_j = 1/n$ and

$$m_{i,j} = \ln\left|\frac{(2j - 2i + 1)^2}{(2j - 2i + 3)(2j - 2i - 1)}\right|. \tag{33}$$

Finite Difference Method

- Expanding the summation in (32) and evaluating the resulting equations at $i = 1, 2, ..., n - 1$ generates a system of $n - 1$ linear equations in $n - 1$ unknowns since $\varphi(x_0) = \varphi_0$ is known and $\varphi_n = 0$ according to (2a) and (2b).
- Imposing the boundary condition $\varphi_x(x) = 0$, another unknown can be determined to get $n - 1$ system in $n - 2$ unknowns.
- As a result, any $n - 2$ is therefore sufficient to determine the remaining unknowns.

Finite Difference Method



(a) $n = 5$ **(b)** $n = 50$ **(c)** $n = 100$

Figure 1: Graph of $\varphi(x)$ plotted against x for (a) $n = 5$, (b) $n = 50$ and (c) $n = 100$, using the conventional finite difference approach.

Integral Approximation Approach

Integral Approximation Approach

In this section we are solving the integral in (1) using a simple approximation method. We divide the interval into equal subintervals and apply an integration rule into each subinterval. Using the left-end point rule we have,

$$\int_{\xi_j}^{\xi_{j+1}} f(\xi, x) = hf(\xi_j, x). \tag{34}$$

Now we implement this technique on our problem. The expression given in (24) can be written as,

$$P_{i\pm\frac{1}{2}} = \int_0^1 \frac{\varphi_\xi(\xi)d\xi}{\xi - \xi_{i\pm,\frac{1}{2}}} = \sum_{j=0}^{n-1} \int_{\xi_j}^{\xi_{j+1}} \frac{\varphi_\xi(\xi)}{\xi - \xi_{i\pm\frac{1}{2}}} d\xi \tag{35}$$

Integral Approximation Approach

With the left-end point rule, we have

$$\sum_{j=0}^{n-1}\int_{\xi_j}^{\xi_{j+1}}\frac{\varphi_\xi(\xi)}{\xi-\xi_{i\pm\frac{1}{2}}}d\xi=\sum_{j=0}^{n-1}h\frac{\varphi_\xi(\xi)}{\xi-\xi_{i\pm\frac{1}{2}}}=\sum_{j=0}^{n-1}\frac{\varphi_{j+1}-\varphi_j}{\xi-\xi_{i\pm\frac{1}{2}}}, \tag{36}$$

which implies that

$$P_{i\pm\frac{1}{2}}=\sum_{j=0}^{n-1}\frac{\varphi_{j+1}-\varphi_j}{\xi-\xi_{i\pm\frac{1}{2}}}. \tag{37}$$

Substituting (37) into (25) we get

$$\sum_{j=0}^{n-1}\frac{\varphi_{j+1}-\varphi_j}{\xi_j-\xi_{i+\frac{1}{2}}}-\sum_{j=0}^{n-1}\frac{\varphi_{j+1}-\varphi_j}{\xi_j-\xi_{i-\frac{1}{2}}}=\frac{\pi}{n} \tag{38}$$

Integral Approximation Approach

which can be expressed as,

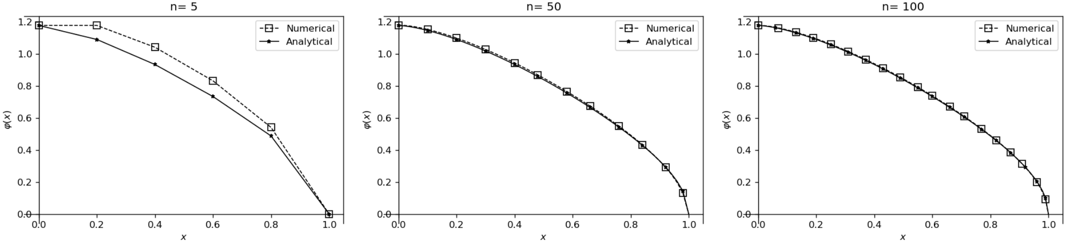
$$\sum_{j=1}^{n-1}(\varphi_{j+1}-\varphi_j)f_{i,j}=\frac{\pi}{2n^2}, \tag{39}$$

where

$$f_{i,j}=\frac{1}{2j-2i-1}-\frac{1}{2j-2i+1} \tag{40}$$

For any choice of n and assuming we know φ_0 and $\varphi_x(0)$, equation (39) will result in a system of $n-1$ equations and $n-2$ unknowns. We will then have a over-determined system and any combination of $n-1$equations can be used to solve for the $n-2$ unknowns.

Integral Approximation Approach



(a) $n = 5$ **(b)** $n = 50$ **(c)** $n = 100$

Figure 2: Graph of $\varphi(x)$ plotted against x for (a) $n = 5$, (b) $n = 50$ and (c) $n = 100$, using the integral approximation approach.

Conclusion

- The two numerical methods performed well in approximating the analytical solution, we unfortunately could not compare the methods using the errors to see which one is is better at approximating the analytical solution.
- There are other more robust numerical methods that could be used such as the linear spline method to solve the given problem.

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